PERFORMANCE ANALYSIS of MODULATED COMMUNICATION SYSTEMS UNDER NOISE
+
RANDOM PROCESSES and SPECTRAL ANALYSIS

References.

Random Processes and Spectral Analysis

- Chapter 9 in the course reference text (Lathi & Ding)
  - Section 9.1: From Random Variable to Random Process.
  - Section 9.2: Classification of Random Processes
  - (*) Section 9.3: Power Spectral Density.
  - (*) Section 9.5: Transmission of Random Processes through Linear Systems.
  - (*) Section 9.7: Performance Analysis of Baseband Analog Systems
  - Section 9.8: Optimum Preemphasis-Deemphasis Systems
  - (*) Section 9.9: Bandpass Random Processes

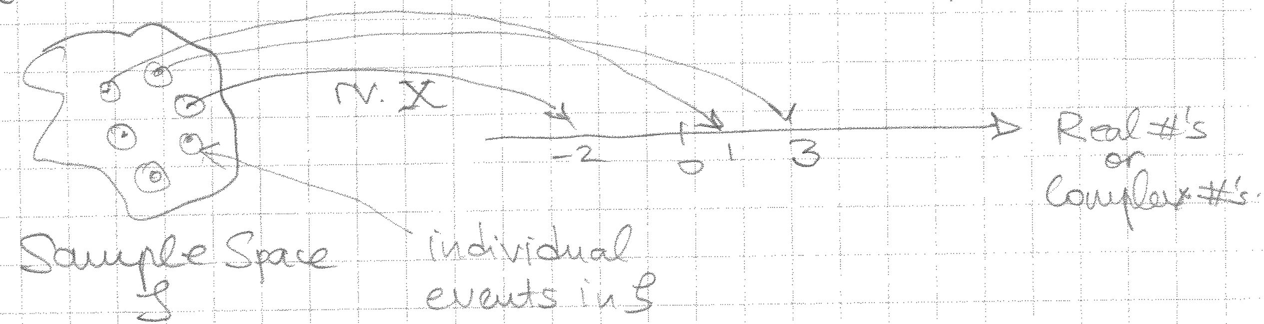
Performance Analysis ....

- Chapter 10 in the course reference text (Lathi & Ding)
  - (*) Section 10.1: Analytical Figure of Merit
  - (*) Section 10.2: Amplitude-Modulated Systems
  - (*) Section 10.3: Angle-Modulated Systems

(*) sections of these two chapters that are critical to our discussion.

Random Variables (RV)

A rv is a mapping from the sample space (usually designated as S or Ω) onto the real/complex numbers



Hence, $\{X(s), s \in S \text{ and } X(s) \in \mathbb{R}\}$ describes the random variable X, we usually supress the notation that shows the dependence of X onto the sample space S and the event $s \in S$ within S. We also associate a probability measure to the rv X. In particular we use the cumulative distribution function $F_X(x)$ to describe the probability of the event.

$$F_X(x) = P[X \le x] = P\{X(s) \le x, \ \forall s \in S\}$$

We work with random variables rather than directly with events and the sample space S because:

(1) it is easier to work with numbers,

(2) if we work with S's directly, then each random phenomenon has to be investigated on its own as an individual random experiment. On the other hand many different random experiments behave in a similar way. Hence, if we study the characteristics of a random variable on such a random experiment, then we have the knowledge about all such random experiments. Hence rv's provide a level of abstraction that allow us efficiently analyze events/phenomena which have random elements.

EX: • $\boxed{\text{Function}}$ — measure initial phase $\Theta_{in} \in [-\pi, \pi]$ uniform

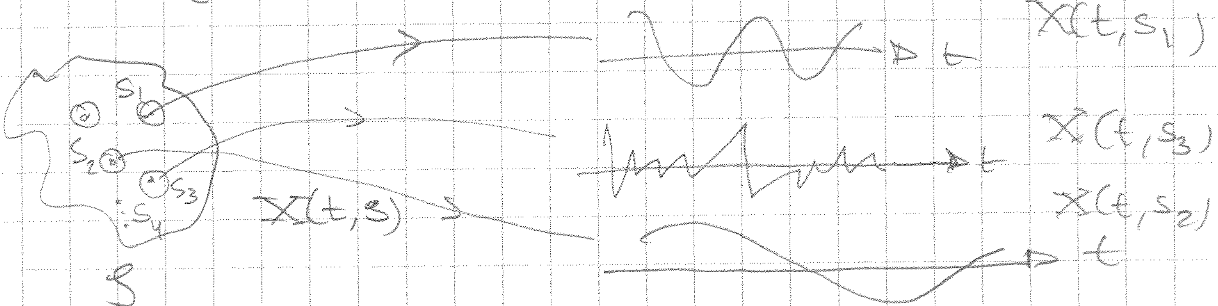
• → ask about ther age $A \in [3, 95]$ 40

• Initial Letter of Last Name $A \to 1$, $Z \to 26$ Letter $= L \in [1, 26]$ uniform

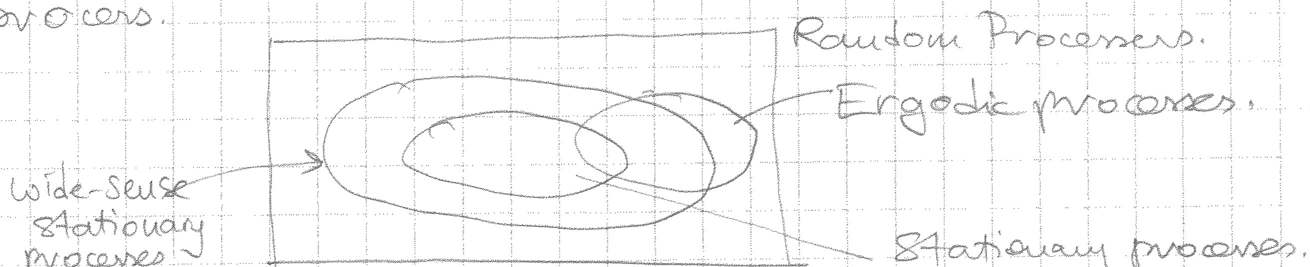
When dealing with rv's we often use statistical averages extracted from the probability distribution of the rv e.g. $E[X]$, $E[X^2]$, σ_X^2 or if we are working with multiple r.v.'s then ρ_{XY}.

Random / Stochastic Processes (RP)

A RP can be consider an extension of a rv, where instead of reaching into a bag (the sample space) and drawing a number at random, the rv, we will reach into the bag / sample space and draw a <u>function of time</u>



Hence, a RP can be described as a collection of functions. An alternate interpretation of a RP is that a RP behaves as a "regular" RV for $t = t^*$. The presence of the time parameter "t" complicates matters considerably. In general it is very difficult to obtain a probabilistic description of an arbitrary RP. Therefore, we usually turn our attention to stationary / wide-sense stationary / ergodic processes.



Each category of RPs is increasingly more well-behaved and therefore are easier to work with.

RPs $\longrightarrow$ Wide-Sense Stationary $\longrightarrow$ Stationary

What are the statistical measures we need when working with RPs?

- Mean value: $E[X(t)] = \mu(t)$

  - For stationary/WSS processes $\mu(t)$ will be independent of time. Hence $\mu(t) = \text{constant}$.

  - For ergodic processes $E[X(t)] = \widetilde{x(t)} = \lim_{T\to\infty} \frac{1}{T}\int_T x(t)\,dt$. That is ensemble and time averages give the same value.

- Autocorrelation function: $E[X(t_1, t_2)] = R_X(t_1, t_2)$

  - For Stationary/WSS processes $R_X(t_1, t_2) = R_X(t_2 - t_1) = R_X(\tau)$.

  - The ACF function is the most important time-domain statistical representation of of a stationary/WSS RP.

  - For an ergodic RP $R_X(\tau) = \widetilde{x(t)x(t+\tau)}$

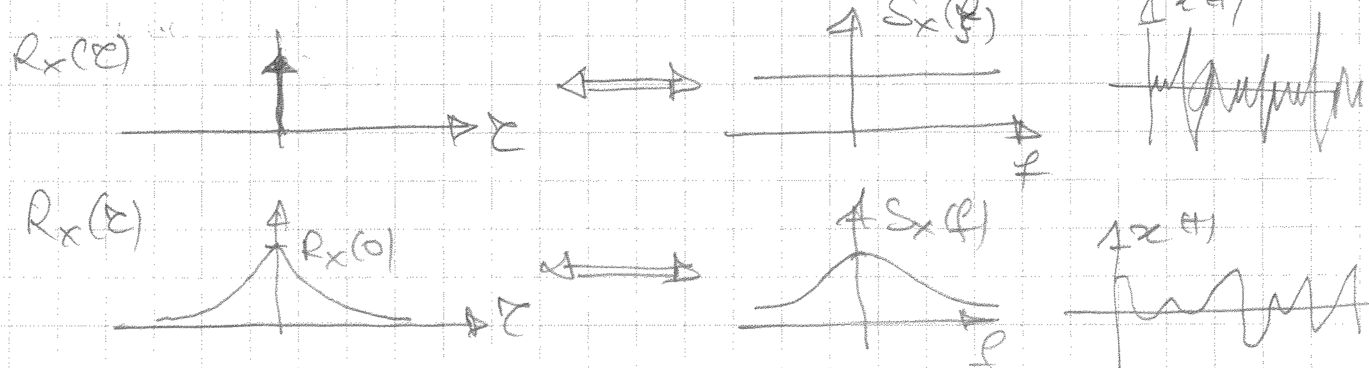
- Power spectral density function $S_X(f)$

$$S_X(f) = \lim_{T\to\infty}\left[\overline{\frac{|X_T(f)|^2}{T}}\right] \quad W/Hz.$$

  - $R_X(\tau) \Longleftrightarrow S_X(f)$ i.e., $S_X(f) = \mathcal{F}[R_X(\tau)]$

  - $P_x = \overline{x^2(t)} = R_X(0) = \int_{-\infty}^{+\infty} S_X(f)\,df$

Observe that



If two RPs $X(t)$ and $Y(t)$ are uncorrelated then

$$R_{XY}(t_1, t_2) = E[X(t_1)Y(t_2)]$$
$$= E[X(t_1)]\,E[Y(t_2)]$$
$$= \bar{x}\;\bar{y} \quad \text{(as } X \text{ and } Y \text{ are wss)}$$

Furthermore, if we consider the sum of two RPs

$$z(t) = x(t) + y(t)$$

then $R_Z(\tau) = R_X(\tau) + R_Y(\tau) + R_{XY}(\tau) + R_{YX}(\tau)$

if x and y are uncorrelated then

$$R_Z(\tau) = R_X(\tau) + R_Y(\tau) + 2\,\bar{x}\,\bar{y}.$$

if $x(t)$ and $y(t)$ are also zero mean processes, that is $\bar{x} = \bar{y} = 0$, then

$$R_Z(\tau) = R_X(\tau) + R_Y(\tau)$$

This results also implies that under the above assumptions

$$S_Z(f) = S_X(f) + S_Y(f)$$

which corresponds to

$$\overline{z^2} = \overline{x^2} + \overline{y^2}$$

i.e., power of the sum process $(z = x + y)$ is the sum of power in $x(t)$, $\overline{x^2}$, and power in $y(t)$, $\overline{y^2}$.

Since communication systems (most of them) consists of linear signal processing operations we also need to understand how the random processes, that are at the input and output of a linear system described by the frequency response function $H(f)$, are related.

- Let the LTI system be described by its impulse response $h(t)$ or equivalently by its frequency response function $H(f)$.
- Let $x(t)$ be a WSS/ergodic random process with ACF $R_x(\tau)$ and PSD $S_x(f)$ be the input to the LTI system
- Let $y(t)$ be the corresponding random process at the output of the LTI system. Let $R_y(\tau)$ and $S_Y(f)$ be the ACF and PSD of $y(t)$ respectively.

$$x(t) \longrightarrow \boxed{H(f)} \longrightarrow y(t)$$

The output ACF is given by

$$R_y(\tau) = h(\tau) * h(-\tau) * R_x(\tau)$$

and therefore

$$S_y(f) = |H(f)|^2 S_x(f)$$

Of course the above result rely on the important result that if $x(t)$ is WSS/ergodic then so is $y(t)$

It is also interesting to compare the above expressions with those with non-random signals i.e.

$$y(t) = h(t) * x(t)$$

$$Y(f) = H(f) X(f).$$

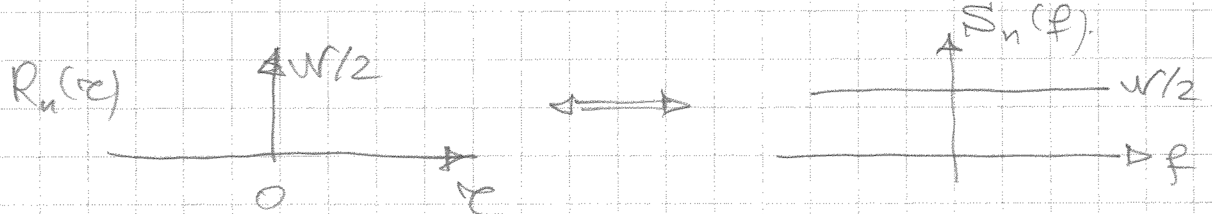
In our discussion of the performance of modulated communication systems, we will measure the performance of such systems under noise. In particular we will be working with the WHITE NOISE PROCESS $n(t)$.

The wideband white noise process is characterized by

$$R_n(\tau) = \frac{N}{2}\delta(\tau)$$

or equivalently by its power spectral density (PSD)

$$S_n(f) = \mathcal{F}[R_n(\tau)] = \frac{N}{2}$$



The above $R_n(\tau)$ indicates that in a wideband white noise process adjacent noise values are uncorrelated, i.e., $E[n(t_1), n(t_1+\tau)] = \emptyset$ no matter how small τ is. The resulting $n(t)$ waveforms will have very random behavior. While such wideband white noise processes are useful idealization/representation of noise processes we encounter in working with communication systems, they are physically unrealizable as they are of infinite power

$$\int_{-\infty}^{+\infty} S_n(f)\,df \rightarrow \infty$$

Therefore, we will work with band-limited/bandpass white noise processes.

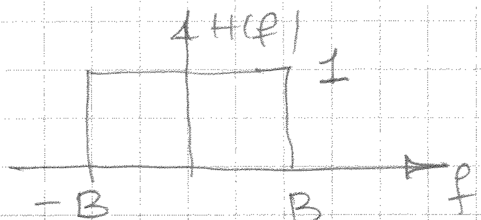
Let's look at some specific examples

Example 1

Determine the ACF, PSD and power of a lowpass noise process

$$n(t) \longrightarrow \boxed{H(f)} \longrightarrow n_{lp}(t)$$

with $H(f)$ representing an ideal lowpass filter.



$$\Rightarrow H(f) = \Pi\left(\frac{f}{2B}\right)$$

Let $n(t)$ be a wideband white noise process with

$$S_n(f) = \frac{\mathcal{N}}{2}$$

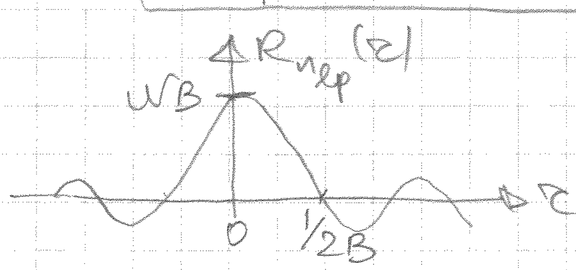
or equivalently, $R_n(\tau) = \mathcal{N}/2\, \delta(\tau)$.

Since $n_{lp}(t)$ is the output process, we have

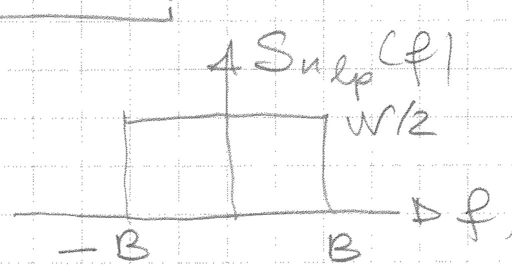
$$S_{n_{lp}}(f) = |H(f)|^2 S_n(f)$$
$$= \left[\Pi\left(\frac{f}{2B}\right)\right]^2 \left(\frac{\mathcal{N}}{2}\right)$$
$$= \boxed{\Pi\left(\frac{f}{2B}\right) \frac{\mathcal{N}}{2}} \quad (\text{as } 1^2 = 1 \text{ for } |f| \le B)$$

and
$$R_{n_{lp}}(\tau) = \mathcal{F}^{-1}\left[\Pi\left(\frac{f}{2B}\right)\frac{\mathcal{N}}{2}\right] = \frac{\mathcal{N}}{2}\, \mathcal{F}^{-1}\left[\Pi\left(\frac{f}{2B}\right)\right]$$
$$= \frac{\mathcal{N}}{2}\, 2B\, \mathrm{sinc}(2\pi B \tau)$$

$$\boxed{R_{n_{lp}}(\tau) = \mathcal{N} B\, \mathrm{sinc}(2\pi B\tau)}$$



and



$$\Rightarrow \overline{n_{lp}^2} = R_{n_{lp}}(0) = \int_{-\infty}^{+\infty} S_{n_{lp}}(f)\, df = \mathcal{N} B \ \mathrm{W}$$

Bandpass Random Processes.

We have shown earlier that a generic bandpass can be formulated as

$$x_{bp}(t) = x_c(t)\cos\omega_c t + x_s(t)\sin\omega_c t$$

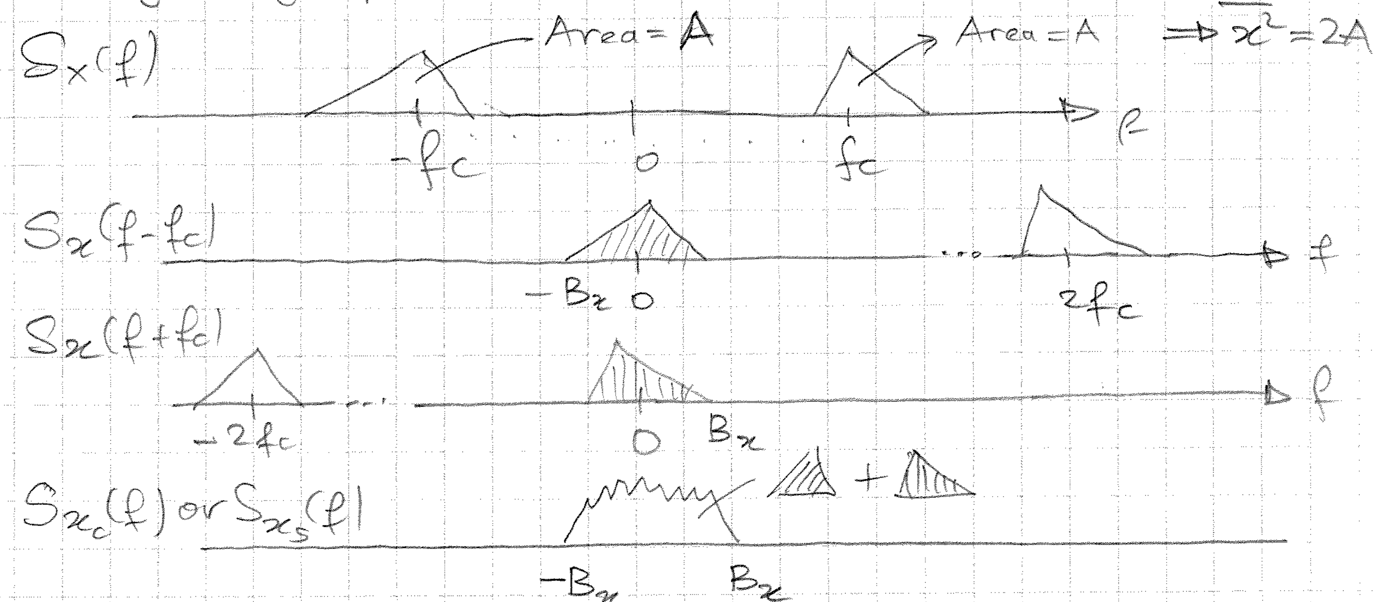
where $x_c(t)$ and $x_s(t)$ are lowpass signal with bandwidth B_x. The resulting bandpass signal will then have the bandwidth $2B_x$ Hz. A bandpass random process can be described in an identical way.

Let $S_x(f)$ be the PSD of a bandpass random process; we can show that

$$S_{x_c}(f) = S_{x_s}(f) = \begin{cases} S_x(f-f_c)+S_x(f+f_c), & |f|\le B_x \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, we can conclude $\overline{x_c^2} = \overline{x_s^2}$.

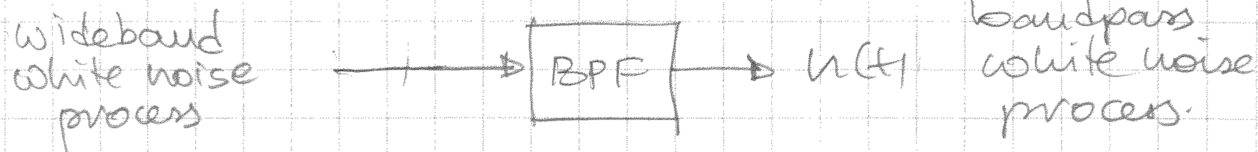
We can show the validity of the above result using a graphical representation of the PSDs.



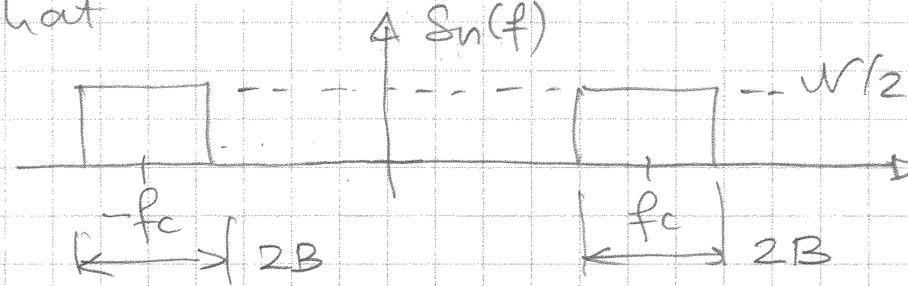
Hence: $\overline{x_c^2} = \int_{-\infty}^{+\infty} S_{x_c}(f)\,df = \int_{-B_x}^{B_x} S_{x_c}(f)\,df = 2A = \overline{x_s^2}$

$$\Rightarrow \overline{x^2} = 2A = \overline{x_c^2} = \overline{x_s^2}$$

Example 2 Let us consider a bandpass white noise which can be generated by bandpass filtering a wideband white noise process.



Such that

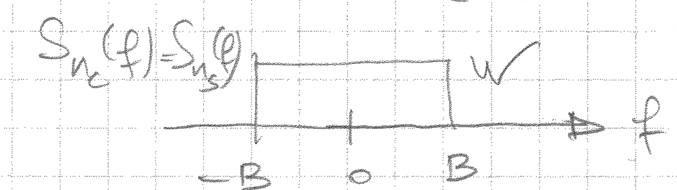


with $n(t) = n_c(t)\cos\omega_c t + n_s(t)\sin\omega_c t$

We want to determine the PSDs of $n_c(t)$ and $n_s(t)$ as well as $\overline{n_c^2}$, $\overline{n_s^2}$ and $\overline{n^2}$.

Since $S_{n_s}(f) = S_{n_c}(f) = \begin{cases} S_n(f-f_c) + S_n(f+f_c), & |f| < B; \\ 0, & \text{otherwise} \end{cases}$

we have



We can now calculate the power in each noise process by calculating the area bounded by its respective PSD:

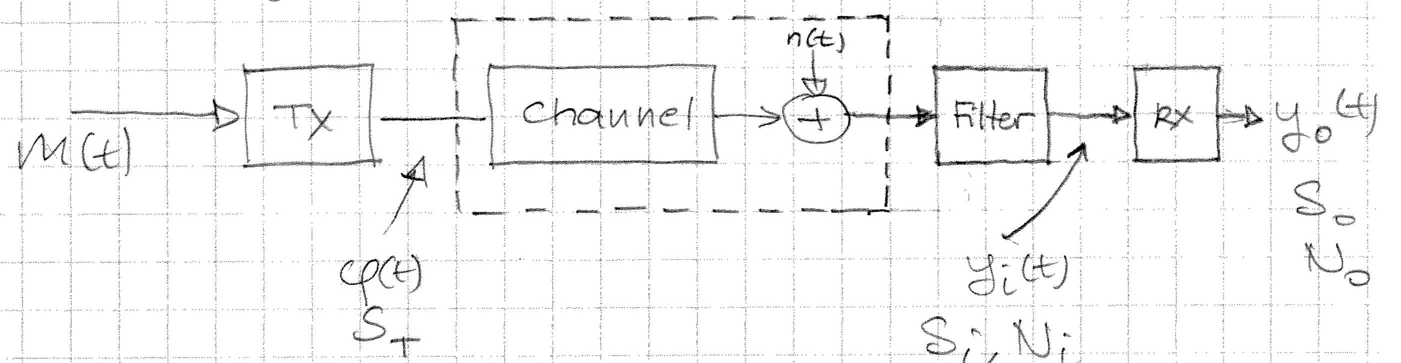
$$\overline{n^2} = \int S_n(f)\,df = (2B)(N/2) + (2B)(N/2) = 2BN$$
$$\overline{n_c^2} = \int S_{n_c}(f)\,df = (2B)N = 2BN = \overline{n_s^2}$$

Observe that if $n(t)$ is a zero-mean WSS process, then:

- $E[n(t)] = E[n_c(t)] = E[n_s(t)] = 0$.
- $n_c(t)$ and $n_s(t)$ are also WSS processes.
- $\overline{n_c^2} = \overline{n_s^2} = \overline{n^2}$
- $\overline{n^2} = \frac{1}{2}\overline{n_c^2} + \frac{1}{2}\overline{n_s^2}$

Performance Analysis of Modulated Communication Systems Under Noise

Our discussion will be based on the following basic communication system model.



$m(t)$: baseband modulating/message signal with baseband bandwidth B_m-Hz.

$\varphi(t)$: modulated waveform with transmission bandwidth B_T-Hz.

S_T: transmitted signal power

$n(t)$: wideband white noise process

$y_i(t)$: noise-corrupted received signal at the output of front-end filter.

S_i: received signal power

N_i: noise power (we will not calculate this directly, we will soon see why).

$y_o(t)$: output of the receiver, includes the demodulated signal and noise.

S_o : signal power at the receiver output.

N_o : noise power at the receiver output.

Furthermore, we will make the following assumptions regarding the above communication system model.

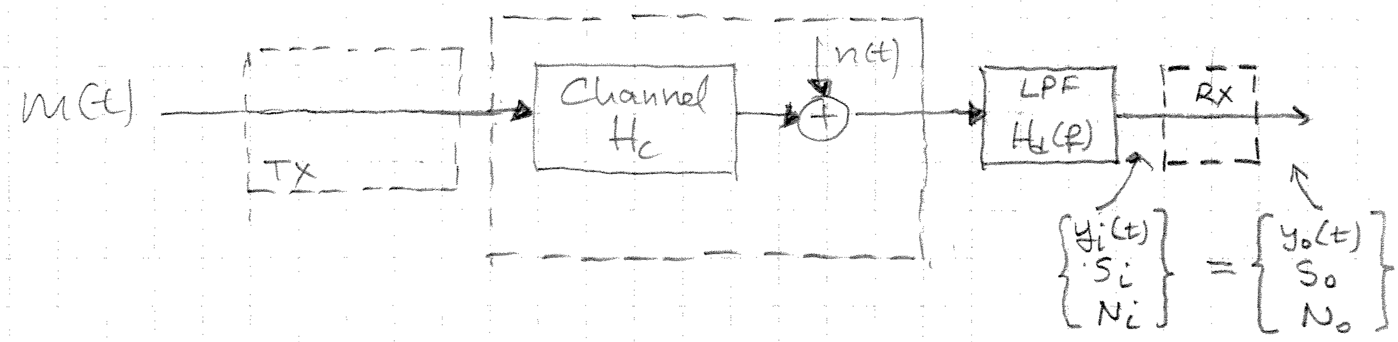
- The channel can and will attenuate and distort the signal $\varphi(t)$.
- All attenuation will be compensated and all distortion will be equalized such that we can assume that the signal component of $y_i(t)$ will be $\varphi(t)$.
- The noise is assumed to be additive.
- In analog communication systems the quality of the received signal is measured in terms of the output signal-to-noise-ratio:

$$SNR_o = S_o / N_o$$

- In general SNR_o can be increased by increasing the transmitted signal power S_T. However, there are other considerations that will limit S_T: limits imposed by regulatory agencies, e.g. CRTC, FCC, channel capability/capacity, signal interference, cost of transmitter ...
- Therefore, our objective will be to calculate SNR_o for a given S_T.
- Also, it may be more convenient to measure SNR_o as a function of S_i (which is of course directly related to S_T).

We will now start to discuss the so-called baseband case to answer the question "What would have been SNR_o if we decided to transmit $m(t)$ is baseband format without using any modulation?"

Baseband System



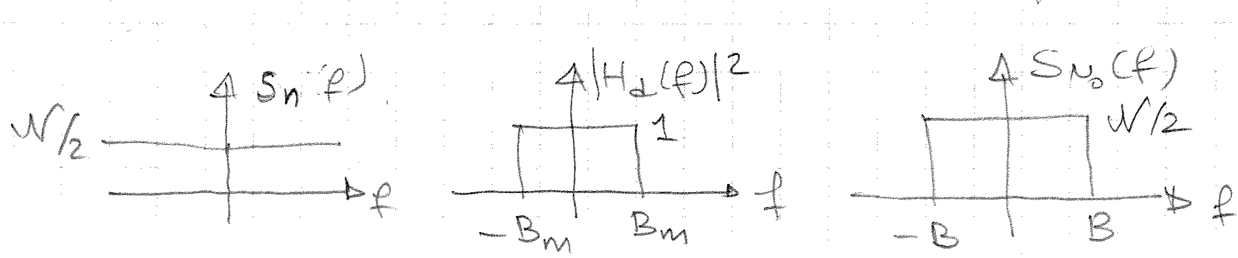
- Channel: no transmission Loss (compensated) distortortionless transmission (equalized).
- Channel input: $m(t)$ with bandwidth B_m-Hz and power $S_m = \overline{m^2(t)}$.
- Channel output: $\sim m(t)$
- Channel noise: $n(t)$ is a white noise process with $S_n(f) = \mathcal{N}/2$.
- Received signal, signal at the receiver input after the lowpass filter:

$$y_i(t) = m(t) + n_i(t)$$

where $S_i = \overline{m^2} = S_m$ as the lowpass filter $H_d(f)$ is compatible with the baseband bandwidth B_m of $m(t)$, and $n_i(t)$ is a lowpass noise process. Furthmore, $S_i = \overline{m^2} = S_m$. It then follows:

$$\begin{aligned} y_o(t) &= y_i(t) \\ &= m(t) + n_o(t) \end{aligned}$$

with $S_o = S_i = \overline{m^2} = S_m$ and $n_o(t) = n_i(t)$ such that $N_o = N_i$. The PSD of $n_i(t) = n_o(t)$ equals:



$$N_o = \int_{-\infty}^{+\infty} S_{N_o}(f)\,df = \int_{-B_m}^{B_m} \mathcal{N}/2\,df = \frac{\mathcal{N}}{2} 2B_m = \mathcal{N}B_m \text{ [W]}$$

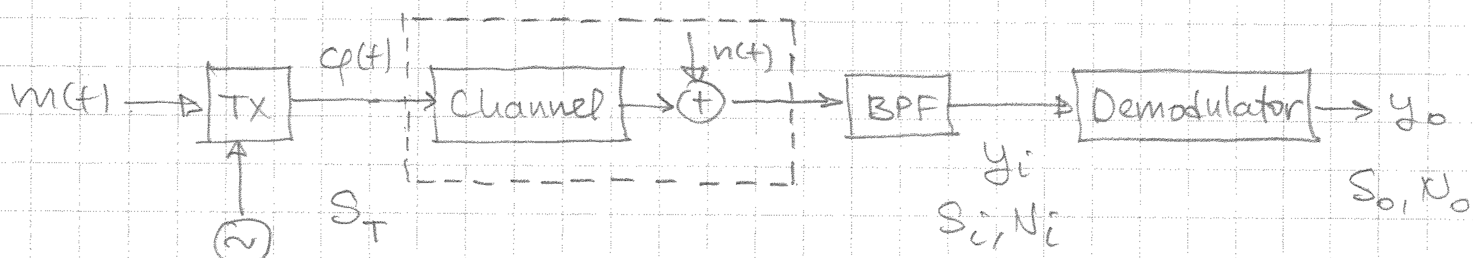
Therefore,

$$\boxed{SNR_o = \frac{S_o}{N_o} = \frac{S_i}{\mathcal{N}B_m} = \gamma}$$

Observe that $S_o = S_i$ which is in turn proportional to S_T.

We will use $\gamma = SNR_o$ (baseband) as a benchmark against other transmission / modulation systems

Bandpass Systems



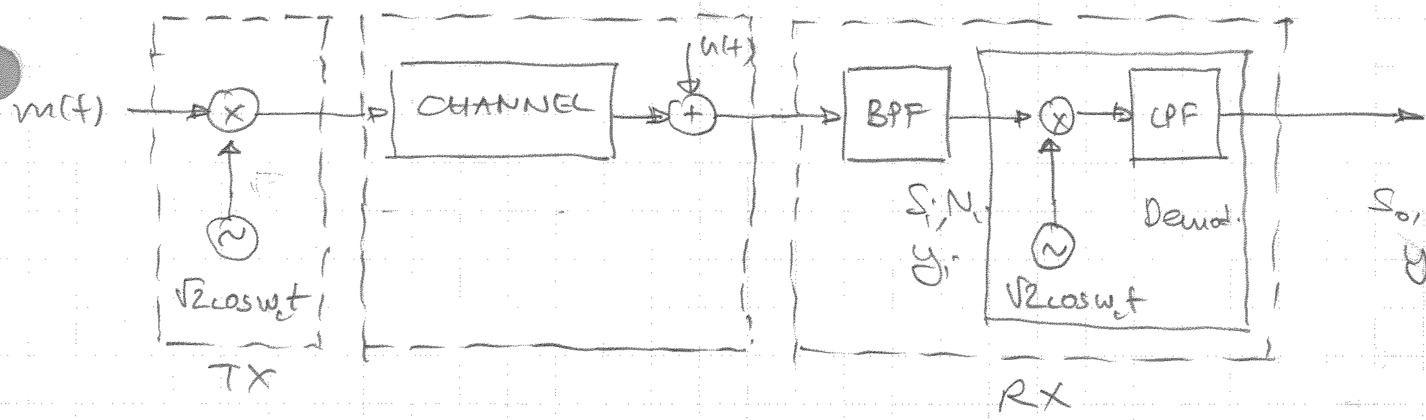
- Let $\varphi(t) = A(t)\cos(w_c t + \phi(t))$ be the modulated waveform
- Channel: Same assumptions, i.e. compensated and equalized.
- BPF output: $y_i(t) = \varphi(t) + n_i(t)$

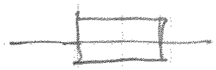
$$= \varphi(t) + [n_c(t)\cos w_c t + n_s(t)\sin w_c t]$$

where the BPF bandwidth is chosen commeasurate to the modulation system used. We will measure the performance of this communication system using a 3 step procedure

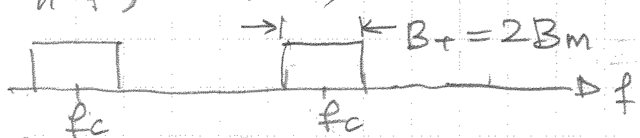
Step ϕ	determine expressions for $y_i(t), n_i(t), y_o(t), n_o(t)$.
Step 1	measure $S_i, N_i \Longrightarrow SNR_i = S_i/N_i$
Step 2	measure $S_o, N_o \Longrightarrow SNR_o = S_o/N_o$
Step 3	Express SNR_o in terms of S_i, SNR_i and compare with the baseband case γ.

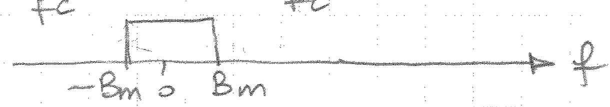
1. DSB-SC



Channel:  compensated, equalized.

Noise: $S_n(f) = W/2$, white noise process

BPF: 

LPF: 

Remark: We use $\sqrt{2}\cos w_c t$ to ensure that $(S_i)_{DSB} = (S_i)_{baseband}$.

Note
$(S_i)_{baseband} = \overline{m^2}$
$(S_i)_{DSB} = \overline{(\sqrt{2} m \cos)^2}$
$= 2\,\overline{m^2}\,(1/2)$
$= \overline{m^2}$

Channel input: $\sqrt{2}\, m(t)\cos w_c t$
Channel output: $\sqrt{2}\, m(t)\cos w_c t + n(t)$
BPF output: $\sqrt{2}\, m(t)\cos w_c t + n_i(t)$

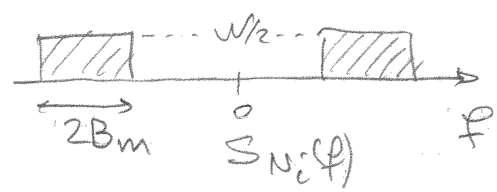
$$y_i(t) = \sqrt{2}\, m(t)\cos w_c t + n_c(t)\cos w_c t + n_s \sin w_c t.$$

Demodulator output: $y_o(t) = \left[\sqrt{2}\, y_i(t)\cos w_c t\right] * h_{LPF}(t)$

$$y_o(t) = \left[2 m(t)\cos^2 w_c t + \sqrt{2}\, n_c(t)\cos^2 w_c t + \sqrt{2}\, n_s(t)\sin w_c t \cos w_c t\right] * h_{LPF}$$

$$\boxed{y_o(t) = m(t) + \frac{\sqrt{2}}{2} n_c(t)}$$

Step 1: $S_i = (\sqrt{2})^2\, \overline{m^2}\, (1/2) = \boxed{\overline{m^2(t)} = S_m}$

N_i: 

$N_i = 2(2B_m)\frac{W}{2} = \boxed{2WB_m}$

Step 2 $S_o = \overline{m^2(t)} = S_m = S_i$

$$N_o = \left(\frac{\sqrt{2}}{2}\right)^2 \overline{n_c^2(t)} = \frac{1}{2}\,\overline{n_c^2(t)}$$

$$= \frac{1}{2}(2\mathcal{N}B) = \boxed{\mathcal{N}B} \quad \text{as } \overline{n^2} = \overline{n_c^2} = \overline{n_s^2}$$

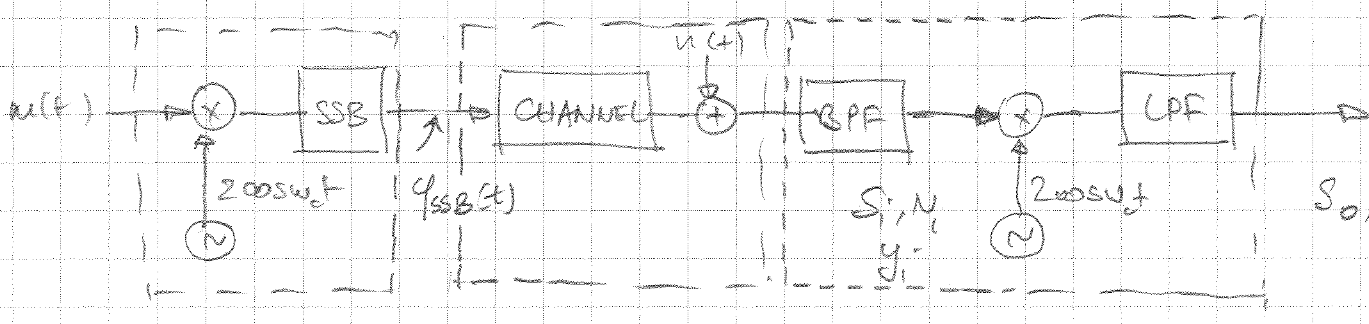
Step 3 $SNR_o = S_o/N_o = S_i/\mathcal{N}B = \boxed{\gamma}$

Observations

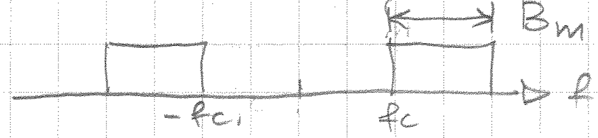
1. Insofar as noise is concerned, DSB-SC with coherent detection has the same performance as baseband transmission.

2. In view of pre-detection noise (N_i) we might have suspected a different result. However, the signal sidebands add coherently when demodulated from f_c to baseband, while noise sidebands add incoherently. The sideband coherence in synchronous demodulation counterbalances the double sideband noise power passed by the BPF.

2. SSB-SC

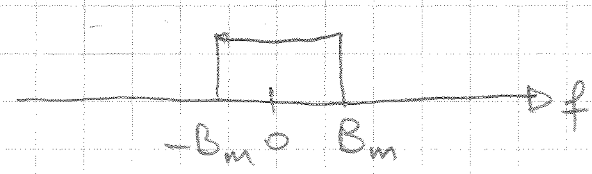


Channel : same
Noise : same
BPF :



(assume USB is retained in φ_{SSB})

LPF :



Channel Input: $\varphi_{SSB} = m(t)\cos w_c t - m_h(t)\sin w_c t$ (USB)

$$\overline{\varphi_{SSB}^2} = \frac{1}{2}\overline{m^2(t)} + \frac{1}{2}\overline{m_h^2(t)} = \overline{m^2(t)} = S_m$$

Channel Output: $y(t) = \varphi_{SSB}(t) + n(t)$

BPF Output: $y_i(t) = \varphi_{SSB}(t) + n_i(t)$

$$y_i(t) = [m\cos w_c t - m_h \sin w_c t] + [n_c \cos w_c t + n_s \sin w_c t]$$

Demodulator Output: $y_o(t) = [2y_i(t)\cos w_c t] * h_{LPF}(t)$

$$y_o(t) = \{[m+n_c]\cos + [n_s - m_h]\sin\}\{2\cos\} * h_{LPF}$$

$$= \{[m+n_c] + [m+n_c]\cos 2w_c t + [n_s - m_h]\sin 2w_c t\} * h_{LPF}$$

$$= m(t) + n_c(t)$$

Step 1: From $y_i = \varphi_{SSB} + n_i \Rightarrow S_i = \overline{\varphi_{SSB}^2} = \overline{m^2} = S_m$

$$N_i = \overline{n_i^2} = \mathcal{N}B$$

Step 2: From $y_o = m + n_c \Rightarrow S_o = \overline{m^2} = S_m$

$$N_o = \overline{n_c^2} = \mathcal{N}B$$

Step 3: $SNR_o = S_o/N_o = S_m/\mathcal{N}B = S_i/\mathcal{N}B = \boxed{\gamma}$

Observations:

1. SSB detection is a straightforward frequency translation. All the signal and frequency components are simply translated (in an unmodified format) to baseband frequencies. Therefore, the relationship between the signal and noise components remains unchanged by SSB modulation/detection.

2. Perf $= SNR_o \Rightarrow SNR_o(\text{baseband}) = SNR_o(\text{DSB-SC}) = SNR_o(\text{SSB-SC})$

3. AMPLITUDE MODULATION

3a. Coherent / Synchronous AM

Same set-up as the DSB-SC case, the transmitted signal be given as.

$$\varphi_{AM}(t) = \sqrt{2}\,(A + m(t)) \cos w_c t.$$

The demolation will remain the same. (except an additional DC block)

Assumptions: same as DSB-SC

Step Ø Channel input: $\varphi_{AM} = \sqrt{2}(A + m(t)) \cos w_c t$

Channel output: $\varphi_{AM} + n = \sqrt{2}(A + m(t)) \cos w_c t + n(t)$

BPF output: $y_i = \sqrt{2}(A + m(t)) \cos w_c t + n_i(t)$

Demodulator o/p: $y_o = [y_i][\sqrt{2} \cos w_c t] * h_{LPF}$

$$y_o'(t) = \left\{2(A+m)\cos^2 + \sqrt{2}(n_c \cos + n_s \sin)\cos\right\} * h_{LPF}$$

$$= \left\{(A+m) + (A+m)\cos 2w_c t + \frac{\sqrt{2}}{2} n_c + (\text{terms } @\, 2f_c)\right\} * h_{LPF}$$

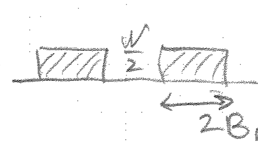
$$= [A + m(t)] + \frac{1}{\sqrt{2}} n_c(t)$$

After DC-block $\quad y_o(t) = m(t) + \frac{1}{\sqrt{2}} n_c(t).$

Step 1 From $y_i \Rightarrow S_i = \overline{[\sqrt{2}]^2 [A+m]^2} \, \overline{\cos^2 w_c t} =$

$$= 2\left[A^2 + 2A\overline{m} + \overline{m^2}\right] \tfrac{1}{2}$$

$$= A^2 + \overline{m^2} \quad (\text{since } \overline{m(t)} = \emptyset).$$

$N_i = \overline{n_i^2(t)} =$ 

$= 2\left(\frac{W}{2}\right) 2B_m = 2WB_m$

Step 2: From $y_o \Rightarrow S_o = \overline{m^2(t)}$

$$N_o = \left(\frac{1}{\sqrt{2}}\right)^2 \overline{n_c^2} = \frac{1}{2}\overline{n_i^2} = \frac{1}{2} 2WB_m = WB_m$$

Step 3: $SNR_i = S_i / N_i = \dfrac{A^2 + \overline{m^2}}{2WB_m}$

and $SNR_o = \frac{\overline{m^2}}{\mathcal{N}B_m} = \frac{\overline{m^2}}{A^2+\overline{m^2}} \frac{A^2+\overline{m^2}}{\mathcal{N}B_m} = \frac{\overline{m^2}}{A^2+\overline{m^2}} \frac{S_i}{\mathcal{N}B_m}$

$$= \boxed{\frac{\overline{m^2}}{A^2+\overline{m^2}} \gamma}$$

Step 4

Observe that $A \geq m_p = \max[m(t)]$ and to maximize SNR_o we need $m_p = A$ (100% modulation)

$$[SNR_o]_{MAX} = \frac{\overline{m^2}}{m_p^2 + \overline{m^2}} \gamma = \frac{1}{\left(\frac{m_p^2}{\overline{m^2}}\right)+1} \gamma$$

Since $m_p^2 \geq \overline{m^2}$, we have $[m_p^2/\overline{m^2}] \geq 1$, and therefore,

$$\boxed{SNR_o \leq [SNR_o]_{MAX} \leq \gamma/2}$$

We can therefore conclude

$$SNR_{o(AM-coh)} \leq SNR_{o(DSB/SSB)} - 3dB.$$

typically $SNR_{o(AM-coh)}$ will be 6dB or more worse than what can be achieved with DSB/SSB.

3b. Envelope Detection

Assumptions: same as before.

Step Ø

Channel input: $\varphi_{AM}(t) = [A+m(t)]\cos\omega_c t$

REMARK: Why $[A+m]$ and not $K[A+m]$ ...
$[A+m] \rightarrow \boxed{ED} \rightarrow \boxed{\text{DC block}} \rightarrow m(t)$ such that $S_o = \overline{m^2} = S_i$

Channel output: $\varphi_{AM+n} = [A+m]\cos\omega_c t + n(t)$.

BPF output: $y_i = [A+m]\cos\omega_c t + n_i(t)$

$= [A+m]\cos\omega_c t + n_c\cos\omega_c t + n_s\sin\omega_c t.$

$= [A+m+n_c]\cos\omega_c t + [n_s]\sin\omega_c t.$

$= \sqrt{[A+m+n_c]^2 + [n_s]^2}\cos(\omega_c t + \theta_E)$

we need the last form for $y_i(t)$ as the demodulator (an envelope detector) will extract the envelope $E(t)$ of $y_i(t) = E(t)\cos(\omega_c t + \theta_E(t))$

Step 1 From $y_i \Rightarrow S_i = \overline{[A+m]^2}\ \overline{\cos^2\omega_c t} = \boxed{\frac{A^2+\overline{m^2}}{2}}$

$N_i = 2NB_m$ as before.

Step 2 First we have consider what $E(t)$ will be equal to, under the following assumptions.

CASE 1 Small noise: $[A+m] \gg n(t)$

$\Rightarrow [A+m] \gg n_c$ or n_s

Hence, since we have at the demod. input

$y_i(t) = \sqrt{[A+m+n_c]^2 + [n_s]^2}\cos(\omega_c t + \theta_E)$

The envelope will be given as

$$E_0(t) \approx [A + m(t) + n_c(t)].$$

Therefore, the demodulator output is

$$y_0(t) = m(t) + n_c(t)$$

From $y_0 \Rightarrow S_0 = \overline{m^2}$ and $N_0 = \overline{n_c^2} = 2NB_m$

CASE 2 Large noise $n \gg [A+m]$.

It can be shown that

$$E(t) = \sqrt{n_c^2 + n_s^2} + [A+m]\cos(-\tan^{-1} n_s/n_c))$$

$E(t)$ contains no terms that are $\sim$ to $m(t)$. $m(t)\cos(-\tan^{-1}\ldots)$ is a noise modulated term. In all previous cases noise was additive. We cannot successfully recover $m(t)$.

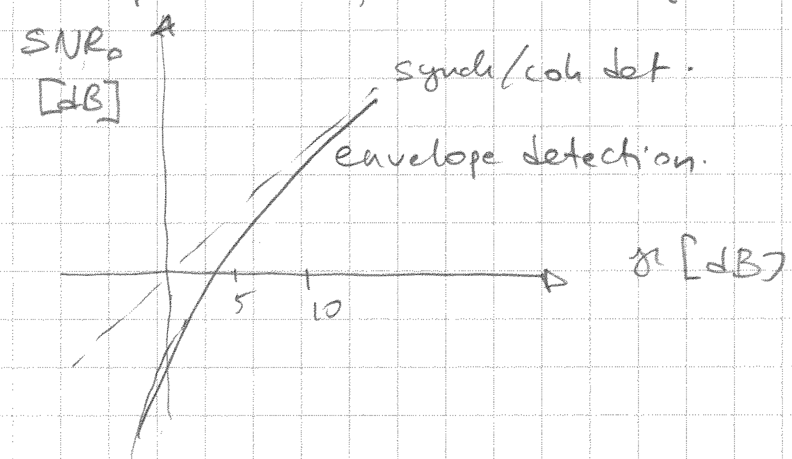
<u>Step 3</u> As under large noise assumption we cannot recover $m(t)$, we will only investigate CASE I — small noise case.

$$SNR_o = \frac{S_o}{N_o} = \frac{\overline{m^2}}{2NB_m} = \frac{\overline{m^2}}{A^2+\overline{m^2}} \; \frac{A^2+\overline{m^2}}{2(NB_m)}$$

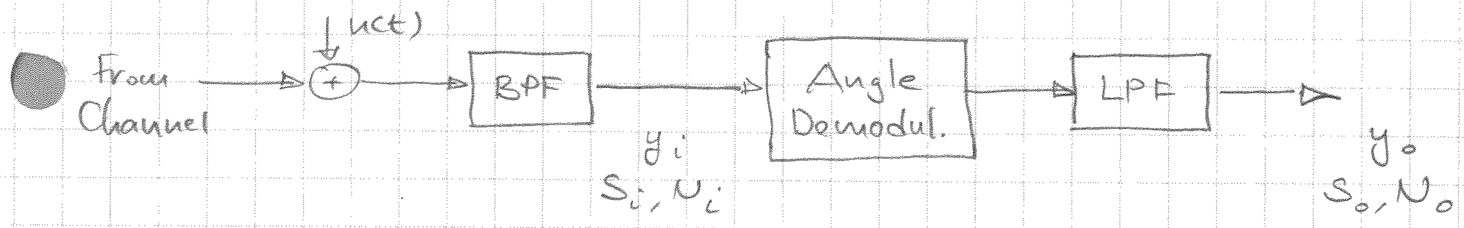
$$= \boxed{\frac{\overline{m^2}}{A^2+\overline{m^2}}\,\gamma}$$

(From book for intermediate case (large – small noise) we can show that $SNR_o \approx 0.92 A^2 \overline{m^2} \gamma$.

<u>Step 4.</u> A comparison of the AM system reveals that:



NOISE ANALYSIS FOR ANGLE DEMODULATED SYSTEMS

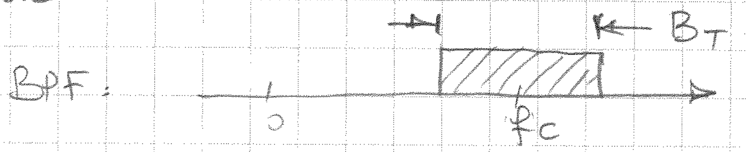


$$y_i(t) = \varphi(t) + n_i(t)$$
$$= A_c \cos(\omega_c t + \Psi(t)) + \left[n_c(t)\cos\omega_c t + n_s(t)\sin\omega_c t\right]$$

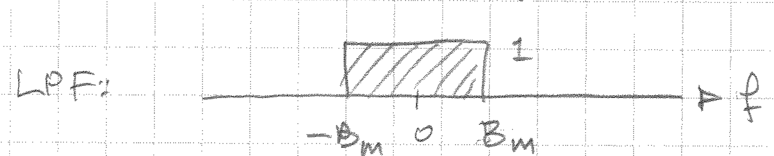
where

$$\varphi(t) = \begin{cases} K_p\, m(t), & \text{PM} \\ K_f \int m(\lambda)\,d\lambda, & \text{FM} \end{cases}$$

and



with $B_T \approx 2\Delta f + 2B_m$



Observe that we can re-formulate $y_i(t)$ and $n_i(t)$ in a number of different ways that are more suitable for further analysis. In particular, let

$$n_i(t) = n_c \cos + n_s \sin = E_n(t)\cos(\omega_c t + \Theta_n(t))$$

$$y_i(t) = A_c \cos(\omega_c t + \Psi(t)) + n_i(t)$$
$$= A_c \cos(\omega_c t + \Psi(t)) + E_n(t)\cos(\omega_c t + \Theta_n(t))$$
$$= R(t)\cos(\omega_c t + \Psi(t) + \Delta\Psi(t))$$

where

$$\Delta\Psi(t) = \tan^{-1}\left\{ \frac{E_n(t)\sin(\Theta_n(t) - \Psi(t))}{A_c + E_n(t)\cos(\Theta_n(t) - \Psi(t))} \right\}$$

For $A_c \gg E_n(t)$ for almost all t, the LOW-NOISE CASE we have

$$A_c + E_n \cos\ldots \approx A_c$$
$$E_n \sin / A_c + E_n \approx \phi \implies \tan^{-1}(\ldots) \approx \ldots$$

Hence

$$\Delta\psi(t) = \frac{E_n(t)}{A_c} \sin\left(\Theta(t) - \psi(t)\right)$$

We also observe that

$$\Delta\psi(t) = \frac{E_n}{A_c}\left[\sin\Theta\cos\psi - \cos\Theta\sin\psi\right]$$

$$= \frac{1}{A_c}\left\{[E_n \sin\Theta]\cos\psi - [E_n\cos\Theta]\sin\psi\right\}$$

$$= \frac{1}{A_c}\left\{[n_s(t)]\cos\psi - [n_c(t)]\sin\psi\right\}$$

Since for a white noise process, its in-phase and quadrature components are incoherent, we can calculate the PSD of the process $\Delta\psi(t)$ as

$$S_{\Delta\psi}(f) = \frac{1}{A_c^2}\left\{S_{n_s}(f)\cos^2\psi(t) + S_{n_c}(f)\sin^2\psi(t)\right\}$$

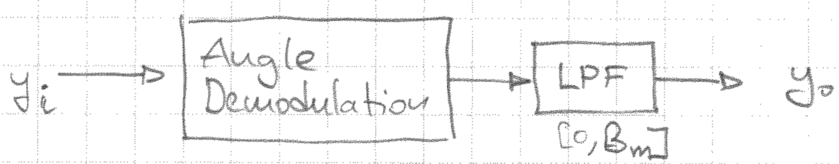
$$= \frac{1}{A_c^2}\left\{\cos^2\psi(t) + \sin^2\psi(t)\right\} S_{n_c}(f)$$

(1.) cross-terms do not appear as n_c and n_s are incoherent.
(2) $S_{n_c}(f) = S_{n_s}(f)$. Hence,

$$S_{\Delta\psi}(f) = S_{n_c}(f)/A_c^2$$

$$S_{\Delta\psi}(f) = \begin{cases} \mathcal{N}/A_c^2, & |f| \le \Delta f + B_m = \frac{B_T}{2}; \\ 0, & \text{otherwise.} \end{cases}$$

Now that we have obtained a time-domain expression we can proceed to analyze the final elements in the receiver structure



AD + LPF: Extracts the time varying phase variation with respect to $\omega_c t$ from $y_i(t)$

Hence

$$y_o(t) = \begin{cases} \psi(t) + \Delta\psi(t), & PM \\ \frac{d}{dt}\left[\psi(t) + \Delta\psi(t)\right], & FM \end{cases}$$

or equivalently

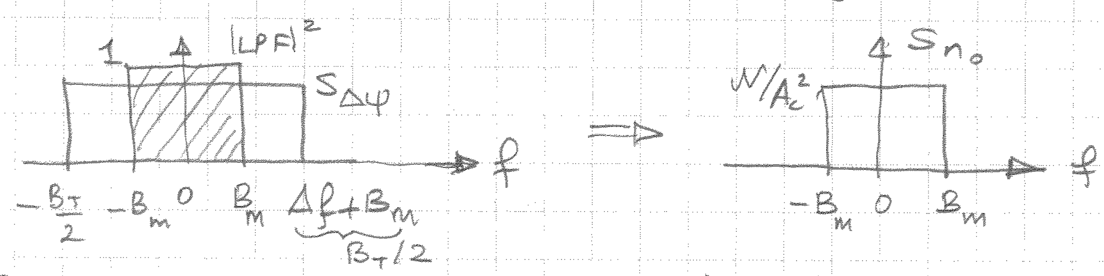
$$S_o(t) = \begin{cases} \psi(t) & \longleftarrow PM \longrightarrow \\ \frac{d}{dt}\psi(t) & \longleftarrow FM \longrightarrow \end{cases} \qquad n_o(t) = \begin{cases} \Delta\psi(t) \\ \frac{d}{dt}\left[\Delta\psi(t)\right] \end{cases} \qquad |f| \le B_m$$

Let us concentrate on the FM case:

$$S_o(t) = K_f\, m(t) \Longrightarrow S_o = \overline{S_o^2} = K_f^2\, \overline{m^2}$$

Also, from $y_i(t) = A_c \cos(\ldots) + n_i(t) \Longrightarrow S_i = A_c^2/2$. The noise process at the output of the AD + LPF is given by $n_o(t)$. Hence

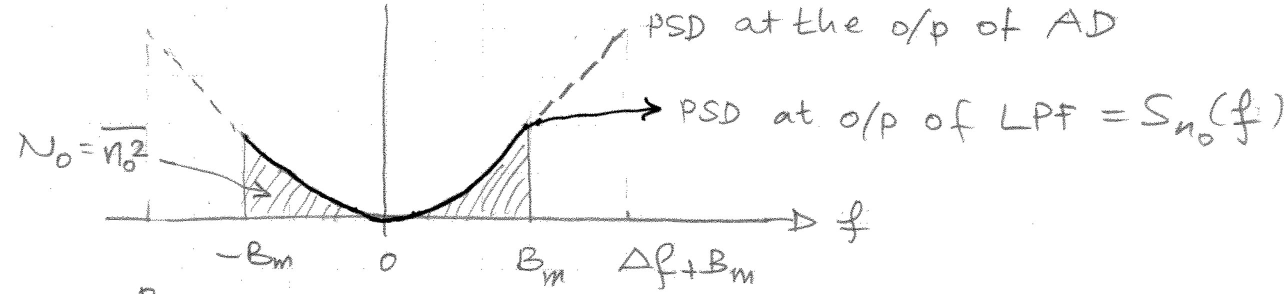
PM:



FM: Since, the output of the AD is $d\psi(t)/dt$

$$\Delta\psi(t) \longleftrightarrow S_{\Delta\psi}(f) = N/A_c^2$$

$$\frac{d}{dt}\Delta\psi(t) \longleftrightarrow |j(2\pi f)|^2 S_{\Delta\psi}(f) = (2\pi f)^2 \frac{N}{A_c^2}$$



$$N_o = \int_{-B_m}^{B_m} S_{n_o}(f)\,df = \int_{-B_m}^{B_m} (2\pi)^2 \frac{\mathcal{N}}{A_c^2} f^2\,df = 4\pi^2 \frac{\mathcal{N}}{A_c^2} \left.\frac{f^3}{3}\right|_{-B_m}^{+B_m}$$

$$= \frac{4\pi^2 \mathcal{N}}{A_c^2}\, 2\frac{B_m^3}{3} \implies \boxed{N_o = \frac{8\pi^2 \mathcal{N} B_m^3}{3A_c^2}}$$

We observe that

$$\gamma = \frac{S_i}{\mathcal{N} B_m} = \frac{A_c^2}{2\mathcal{N} B_m}$$

Such that

$$SNR_o = \frac{S_o}{N_o} = \left[K_f^2\, \overline{m^2}\right]\left[\frac{8\pi^2 B_m^3 \mathcal{N}}{3A_c^2}\right]^{-1}$$

$$= 3\,\frac{K_f^2\, \overline{m^2}}{4\pi^2 B_m^2}\;\frac{A_c^2}{2\mathcal{N} B_m}$$

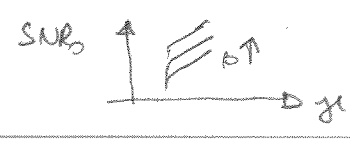
$$= 3\,\frac{K_f^2}{(2\pi B_m)^2}\,\overline{m^2}\,\gamma$$

where $\Delta\omega = K_f m_p$ or $\Delta f = K_f m_p / 2\pi$ and $\beta = \Delta f / B_m$
We can now write the SNR_o as

$$SNR_{o-FM} = 3\beta^2 \frac{\overline{m^2}}{m_p^2}\,\gamma$$

For example, for the single tone case, $m(t) = A_m \cos\omega_m t$ with $m_p = A_m$, $\overline{m^2} = A_m^2/2$ and $\overline{m^2}/m_p^2 = 1/2$

$$SNR_{o-FM} = \tfrac{3}{2}\beta^2\gamma$$



Given that for AM we determined

$$SNR_{o-AM} = \frac{\overline{m^2}}{A_c^2 + \overline{m^2}} \gamma$$

and at maximum allowable (100%) modulation $A_c = m_p$ we can rewrite SNR_{o-AM} as

$$SNR_{o-AM} = \frac{\overline{m^2}}{m_p^2 + \overline{m^2}} \gamma$$

and then compare with SNR_{o-FM}. In particular AM will outperform FM if

$$SNR_{o-FM} \leq SNR_{o-AM}$$

$$3\beta^2 \frac{\overline{m^2}}{m_p^2} \gamma \leq \frac{\overline{m^2}}{m_p^2 + \overline{m^2}} \gamma$$

$$\beta^2 \leq \frac{1}{3} \frac{1}{1 + \frac{\overline{m^2}}{m_p^2}} < 1/3$$

Hence, for

$$\beta^2 < 1/3 \quad (\text{or } \beta < 1/\sqrt{3} \approx 0.6)$$

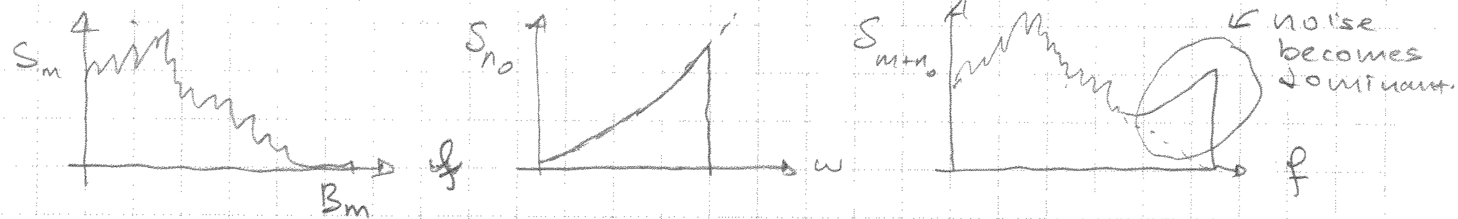
AM performs better than FM

$\Longrightarrow$ NBFM performs similar to AM
WBFM has superior noise performance

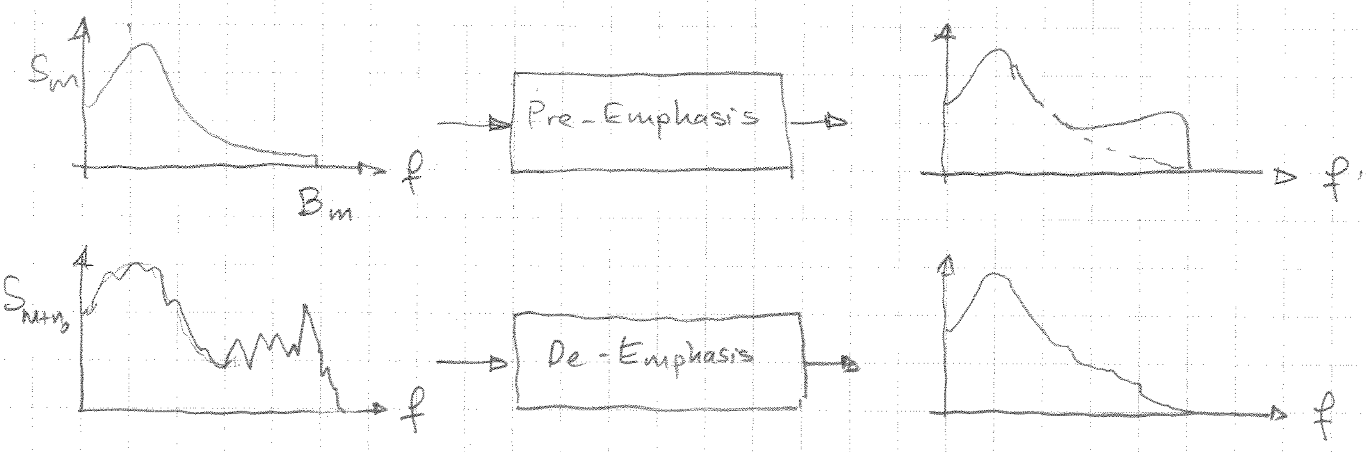
SNR IMPROVEMENT USING PRE- & DE-EMPHASIS

(See Section 5.5 for a discussion)

- At the output of FM demodulator noise power rises quadratically with increasing frequency.
- Worst case scenario: signal power (speech, audio) is lowest at high frequencies.

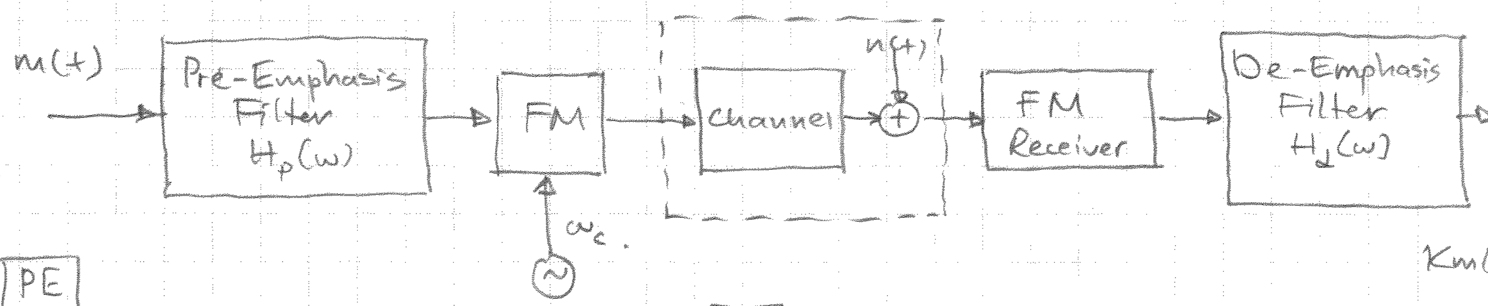


- Therefore, we want to <u>emphasize</u>, i.e., amplify the high frequency components of the input signal at the transmitter before the noise is introduced. (⟹ pre-distortion).



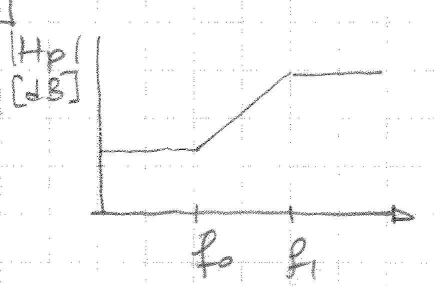
REMARK: not too much pre-emphasis as $B_T \uparrow$
Rule of thumb: PE characteristic should be of a form such that the noise spectral density after [demodulation + DE] should resemble white noise (i.e., flat PSD).

Since $S_{n_0} \sim \omega^2 \Longrightarrow PE \sim \omega^2 \Longrightarrow$ differentiate FM
$\Longrightarrow$ FM becomes PM at high frequencies.



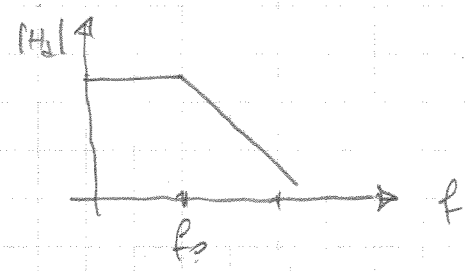
Km(t)

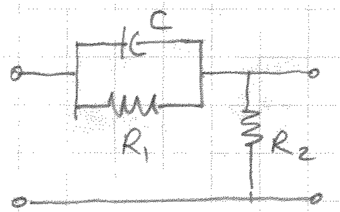
PE

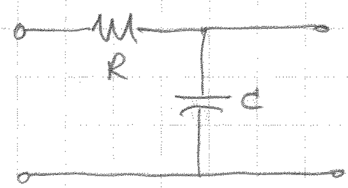


$f_1 > 15$ kHz

DE







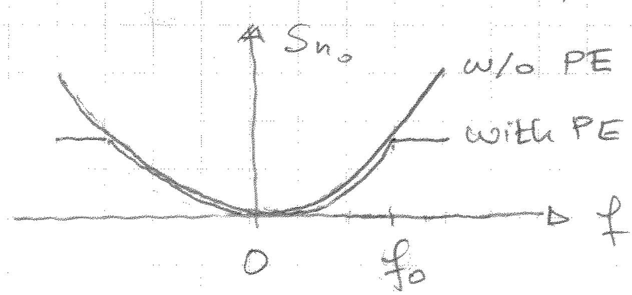
$2\pi f_0 = 1/R_1C \quad 2\pi f_1 = 1/R_pC \qquad 2\pi f_0 = 1/RC$

- For North American broadcasting systems

$$f_0 = 2.1 \text{ kHz} \Longrightarrow RC = 75\,\mu s.$$

- For European broadcasting system

$$f_0 = 1.275 \text{ kHz} \Longrightarrow RC = 125\,\mu s.$$



Let us recall

$$S_{n_o}(f) = \begin{cases} \frac{N}{A_c^2}(2\pi f)^2, & |f| \le B_T/2 \\ 0, & \text{otherwise.} \end{cases}$$

and

$$N_o = \int_{-B_m}^{B_m} S_{n_o}(f)\,df = \frac{8\pi^2 N}{3A_c^2} B_m^3$$

with PE + DE

$$n_o' \longrightarrow \boxed{\begin{array}{c}\text{DE}\\ H_d(\omega)\end{array}} \longrightarrow n_o'$$

$$S_{n_o'}(f) = |H_d(f)|^2 S_{n_o}(f), \text{ with}$$

$$N_o' = \int_{-B_m}^{+B_m} S_{n_o'}(f)\, df$$

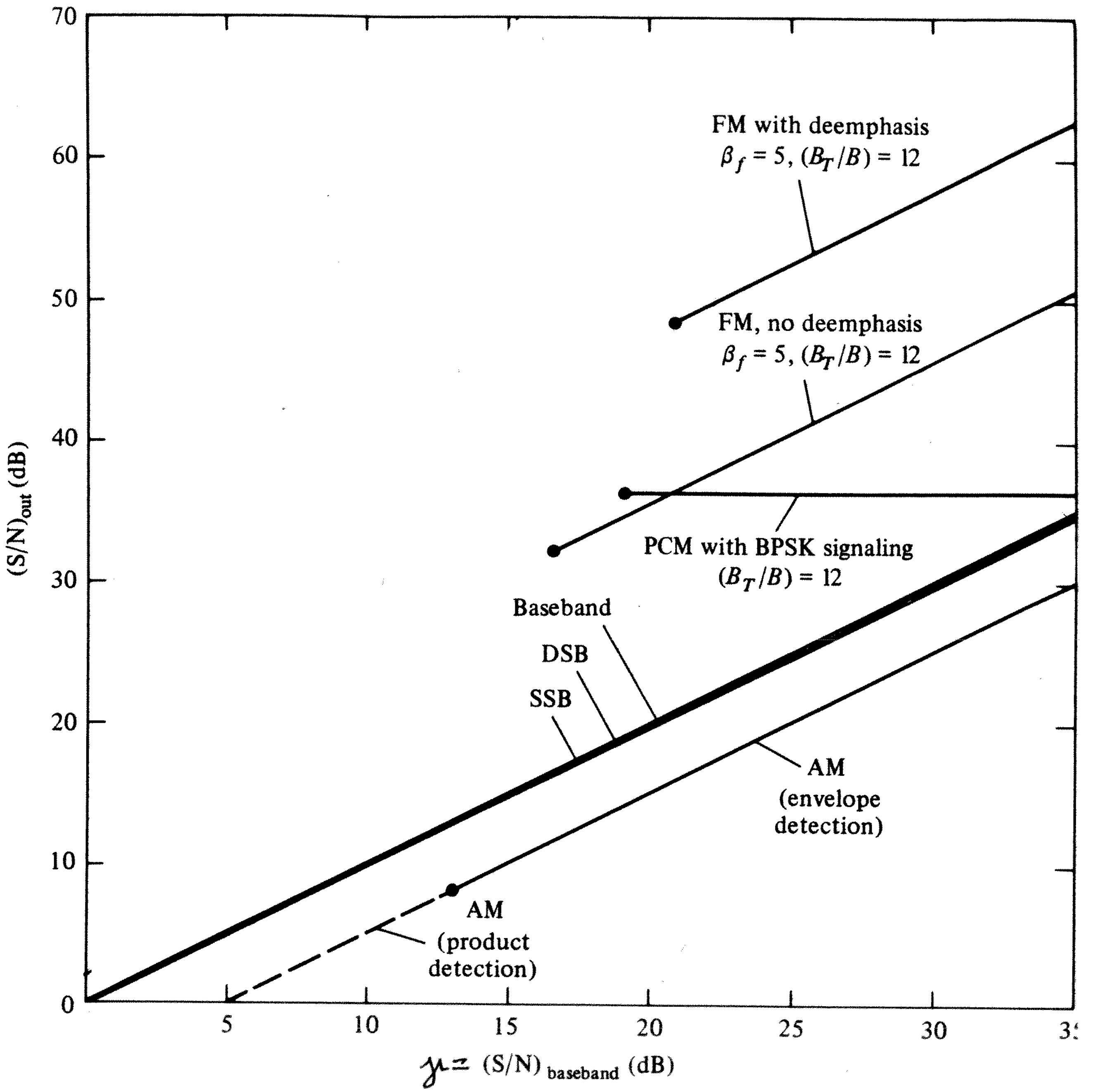
$$= \int_{-B_m}^{B_m} |H_{DE}(f)|^2 \frac{\mathcal{N}}{A_c^2}(2\pi f)^2\, df$$

$$= \frac{\mathcal{N}}{A_c^2} \int_{-B_m}^{B_m} \frac{(2\pi f)^2}{1+(f/f_o)^2}\, df = \dots$$

The improvement in SNR_o

$$I = \frac{S_{n_o'}}{S_{n_o}} = \frac{(B/f_o)^3}{3(B/f_o) - \tan^{-1}(B/f_o)}$$

For $f_o = 2.1$ kHz $\Rightarrow I \approx 18 \Rightarrow$ 13 dB improvement in SNR_o.



Comparison of the noise performance of analog systems.